VIII.—*On the Equilibrium and Motion of an Elastic Solid.* By the Rev. JOHN H. JELLETT, Fellow of Trinity College, and Professor of Natural Philosophy in the University of Dublin.

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1. THE problem which forms the subject of the present Memoir has already, at various times, occupied the attention of mathematicians. Although much of the interest which it has excited is due to its connexion with the undulatory theory of light, the importance of the problem itself, considered as a branch of rational mechanics, is fully admitted; and more than one writer has treated of it without regard to the real or supposed existence of a luminous ether. Nor can it, I think, be doubted, that such a distinction between the rational and the physical science, is in accordance with the dictates of just philosophy. The rational science would still be real, even though the existence of the ether were (if that were possible) disproved; and the admitted reality of the several solid and fluid bodies which are found in nature gives us, in such cases, the means of testing by experiment the accuracy of the laws arrived at. "Whatever theoretic objections," says Mr. HAUGHTON, "may be made to the application of the theory of elastic media to optics, none such exist as to its application to solid and fluid bodies. The mathematical investigations which, in the case of light, must be hypothetical, are, in the case of solid and fluid bodies, essentially positive, and may be made the subject of direct experiment. A general inquiry into the laws of elastic media is an interesting application of rational mechanics; and although it must necessarily include cases purely hypothetical, it is not, therefore, to be considered unimportant."*

* Transactions of the Royal Irish Academy, Vol. xxii. Part i. p. 97.

2. Two general methods have been adopted by the various authors who have treated of this problem. Of these, the one consists in forming expressions for the forces which act upon each particle in the medium under consideration, and then determining the laws of its equilibrium, or motion, by the general statical or dynamical equations. This method is followed by POISSON and CAUCHY. It is also adopted by NAVIER in the commencement of his Memoir, but soon abandoned, as being less complete than the second method. This latter, which is the method of LAGRANGE, and is followed by Mr. GREEN, Professor MAC CULLAGH, and Mr. HAUGHTON,* takes as its basis the equation derived from the combination of D'ALEMBERT's principle with that of virtual velocities, and is distinguished by the greater completeness of the solution which it affords; the same analysis giving both the general equations of equilibrium, or motion, and the particular conditions which must be satisfied at the bounding surface of the body or medium under consideration.† This is the method which I propose to adopt in the present Memoir. The discussion of a problem like the present must, of course, rest upon principles more or less hypothetical, inasmuch as the nature of molecular action cannot (at least in the present state of physical knowledge) be ascertained by direct experiment. The classification, however, with which the present investigation commences, cannot be considered as other than positive, inasmuch as the two kinds of force, upon the distinction between which it is founded, are known to exist in nature, and cannot, without a hypothesis, be reduced to one. The principle of this classification I shall now proceed to state.

* All these writers commence with the assumption that the sum of the internal moments of a medium may be represented by the variation of a single function. To this method it may, perhaps, be objected that it takes, as the foundation of a physical theory, a principle which is almost purely mathematical, and to which it appears difficult to give a definite physical meaning. This hypothesis, moreover, does not give to the equations of motion all the generality of which they are susceptible. I have, therefore, preferred taking, as the basis of the present Memoir, a principle essentially physical; more especially as the equations of motion derived from this principle are, in the case of homogeneous bodies, possessed of the full number of constants, and have, therefore, the greatest amount of generality which their form admits.

† The investigation of these conditions, according to the method ordinarily adopted, is, however, open to serious objections. These the reader will find noticed in a subsequent part of the present Memoir.

GENERAL CLASSIFICATION OF BODIES.

I.—*Hypothesis of Independent Action.*

II.—*Hypothesis of Modified Action.*

3. The classification which I propose here to adopt, and which forms the basis of the present Memoir, is founded upon the following very obvious principle. The force, or influence, which one particle or molecule exerts on another, may show its effect either by causing a change in its *state,* or by causing a change in its *position.* Either or both of these changes may affect the influence which this particle in its turn exerts upon any of those around it. Thus, for example, if m, m', m'', be three particles acting upon each other by the ordinary attraction of gravitation, the action of m' upon m'' will be modified by the action of m only so far as their distance from each other is changed by it. The attraction of m has no power to change the attraction of m' upon any other particle, except by altering its distance from that particle. But the case would be altogether different if we supposed m, m', m'', to be *electrified* particles. In this case the action of m upon m' would modify the action of that particle upon m'', not only by changing the distance between them, but also by changing their electrical state, and, therefore, the force which each exerts upon the other. In the former case, if m' and m'' maintain the same relative position, the force which they mutually exert remains unchanged. In the second, even though the relative position of the two particles remains unaltered, their mutual action will be modified by the presence of a third particle.* From this distinction an obvious classification follows. In the first class we place all bodies whose particles exert upon each other a force which is *independent* of the surrounding particles; a force, therefore, which can be changed only by a displacement of one or both of the particles under consideration. In the second class, which includes all other bodies, the mutual action of two particles is supposed to be affected by that of the surrounding particles.

* I do not, of course, mean to say, that in a case like that of electrified particles, change of state in the particle itself may not be caused by change of position in the particles of some fluid which pervades it. It is sufficient for my purpose, that in such a case the force which two particles exert upon each other may be changed without a displacement of the particles themselves.

We shall now proceed to investigate the equations of equilibrium and motion, for bodies of the first class.

I.—HYPOTHESIS OF INDEPENDENT ACTION.

4. Let the several particles of a body, which satisfies the hypothesis of independent action, be displaced from their original position of free equilibrium, this displacement being supposed to follow some regular law. Let it be required to determine the conditions of equilibrium of these particles in their new position, or, in other words, to assign the forces which should be applied to each of them in order to keep them at rest. Again, if the particles be left to themselves after the displacement, let it be required to determine the law of their motion.

In applying the method of LAGRANGE to any problem of equilibrium or motion, it is plainly necessary to commence with two assumptions, namely:— 1. An assumed expression for the *intensity* of each of the acting forces. 2. An assumed expression for the *effect* which this force tends to produce; the effect of a force being defined by the quantity which it tends to change.

Let m, m' be two particles of the body under consideration, and let F be the force which, in their displaced position, they exert upon each other.

Let x, y, z be the co-ordinates of m in its original position, and ξ, η, ζ its resolved displacements. Let also x', y', z', ξ', η', ζ' be the co-ordinates and displacements of m' .Then, since, by the hypothesis of independent action, F does not depend upon the displacement of any of the other particles, and since, if the body have a regular constitution, the state of each particle must be a function of its position,

$$F = f(x, y, z, \quad x', y', z', \quad \xi, \eta, \zeta, \quad \xi', \eta', \zeta');$$

or, as it may be otherwise written,

$$F = f(x, y, z, \quad x', y', z', \quad \xi, \eta, \zeta, \quad \xi' - \xi, \quad \eta' - \eta, \quad \zeta' - \zeta).$$

But in all media with which we are acquainted, no internal force appears to be generated by a mere transference of the entire system from one position in space to another, the *relative* positions of the several particles remaining unchanged.

This being supposed universally true, we shall have, as is easily seen,

$$F=f\,(x,\, y,\, z,\quad x',\, y',\, z',\quad \xi'-\xi,\quad \eta'-\eta,\quad \zeta'-\zeta).$$

Let ρ, θ, ϕ be the polar co-ordinates of m' with regard to m; then since

$$x'=x+\rho\sin\theta\cos\phi,\quad y'=y+\rho\sin\theta\sin\phi,\quad z'=z+\rho\cos\theta\,;$$

it is plain that the foregoing expression for x may be written

$$F=f\,(x,\, y,\, z,\quad \rho,\, \theta,\, \phi,\quad \xi'-\xi,\quad \eta'-\eta,\quad \zeta'-\zeta).$$

Hitherto no assumption has been made either with respect to the magnitude of the distance between the particles m, m', or with respect to that of the displacements ξ, η, ζ, ξ', η', ζ'. But previously to proceeding further, it is necessary to make the following suppositions:

(1.) That the greatest distance between two particles which are capable of acting upon one another, or, as it is ordinarily termed, the radius of molecular activity, is indefinitely small compared with the intensity of the force generated.

(2.) That the sphere of molecular activity contains, nevertheless, an indefinitely great number of particles.

From the first of these assumptions, combined with the supposition that the displacements follow some regular law, we have

$$\begin{aligned}
\xi' &= \xi+\frac{d\xi}{dx}\,dx+\frac{d\xi}{dy}\,dy+\frac{d\xi}{dz}\,dz,\\
\eta' &= \eta+\frac{d\eta}{dx}\,dx+\frac{d\eta}{dy}\,dy+\frac{d\eta}{dz}\,dz,\\
\zeta' &= \zeta+\frac{d\zeta}{dx}\,dx+\frac{d\zeta}{dy}\,dy+\frac{d\zeta}{dz}\,dz.
\end{aligned}\qquad\text{(A)}$$

quantities of higher orders being neglected.

For the same reason,

$$F=F_0+A\,(\xi'-\xi)+B\,(\eta'-\eta)+C\,(\zeta'-\zeta).\qquad\text{(B)}$$

This expression consists, as will be seen, of two distinct parts, namely F_0,

which represents the force which m' exerts upon m in the original position of these particles, and

$$A\,(\xi' - \xi) + B\,(\eta' - \eta) + C\,(\zeta' - \zeta),$$

the force generated by the displacement.

The supposition that the original state of the body was one of free equilibrium permits us to disregard the former of these parts. For it follows from that supposition, that if the several particles of the body receive equal displacements, the new position is also a position of equilibrium. Hence the suppositions,

$$\xi' = \xi, \quad \eta' = \eta, \quad \zeta' = \zeta,$$

must satisfy the general equation of equilibrium. But these suppositions give

$$F = F_0.$$

Hence the terms depending upon F_0 will disappear of themselves. We have, therefore, for the *effective* part of the force,

$$F = A\,(\xi' - \xi) + B\,(\eta' - \eta) + C\,(\zeta' - \zeta),$$

where A, B, C are in general of the form

$$f\,(x, y, z, \ \rho, \theta, \phi).$$

Let α, β, γ be the angles which the direction of ρ makes with the axes, so that

$$\cos\alpha = \sin\theta\cos\phi, \quad \cos\beta = \sin\theta\sin\phi, \quad \cos\gamma = \cos\theta.$$

Then since

$$dx = \rho\cos\alpha, \quad dy = \rho\cos\beta, \quad dz = \rho\cos\gamma,$$

we shall have from equations (A),

$$\begin{aligned}
\xi' - \xi &= \rho\left(\cos\alpha\frac{d\xi}{dx} + \cos\beta\frac{d\xi}{dy} + \cos\gamma\frac{d\xi}{dz}\right),\\
\eta' - \eta &= \rho\left(\cos\alpha\frac{d\eta}{dx} + \cos\beta\frac{d\eta}{dy} + \cos\gamma\frac{d\eta}{dz}\right),\\
\zeta' - \zeta &= \rho\left(\cos\alpha\frac{d\zeta}{dx} + \cos\beta\frac{d\zeta}{dy} + \cos\gamma\frac{d\zeta}{dz}\right),
\end{aligned} \qquad \text{(C)}$$

and therefore,

$$
\begin{aligned}
F = A\rho\left(\cos\alpha\frac{d\xi}{dx} + \cos\beta\frac{d\xi}{dy} + \cos\gamma\frac{d\xi}{dz}\right)& \\
+ B\rho\left(\cos\alpha\frac{d\eta}{dx} + \cos\beta\frac{d\eta}{dy} + \cos\gamma\frac{d\eta}{dz}\right)& \qquad (D) \\
+ C\rho\left(\cos\alpha\frac{d\zeta}{dx} + \cos\beta\frac{d\zeta}{dy} + \cos\gamma\frac{d\zeta}{dz}\right)&.
\end{aligned}
$$

Let α', β', γ' be the angles which the direction of this force makes with the axes, and X, Y, Z its components. Then

$$
\begin{aligned}
X = F\cos\alpha' = \rho\cos\alpha'\Big\{A\left(\cos\alpha\frac{d\xi}{dx} + \cos\beta\frac{d\xi}{dy} + \cos\gamma\frac{d\xi}{dz}\right)& \\
+ B\left(\cos\alpha\frac{d\eta}{dx} + \cos\beta\frac{d\eta}{dy} + \cos\gamma\frac{d\eta}{dz}\right)& \\
+ C\left(\cos\alpha\frac{d\zeta}{dx} + \cos\beta\frac{d\zeta}{dy} + \cos\gamma\frac{d\zeta}{dz}\right)\Big\}&; \qquad (E)
\end{aligned}
$$

$$
Y = F\cos\beta' = \rho\cos\beta'\left\{A\left(\cos\alpha\frac{d\xi}{dx} + \&c.\right) + \&c.\right\};
$$

$$
Z = F\cos\gamma' = \rho\cos\gamma'\left\{A\left(\cos\alpha\frac{d\xi}{dx} + \&c.\right) + \&c.\right\}.
$$

We have next to consider the effect which this force tends to produce; and on this point the assumption here made is, *that the forces developed by the displacements of the several particles tend to change their relative positions only.* Hence it is evident, that the moments of the forces X, Y, Z will be

$$
X\delta(\xi' - \xi), \quad Y\delta(\eta' - \eta), \quad Z\delta(\zeta' - \zeta),
$$

respectively, or

$$
\begin{aligned}
&\rho X\left(\cos\alpha\frac{d\delta\xi}{dx} + \cos\beta\frac{d\delta\xi}{dy} + \cos\gamma\frac{d\delta\xi}{dz}\right), \\
&\rho Y\left(\cos\alpha\frac{d\delta\eta}{dx} + \cos\beta\frac{d\delta\eta}{dy} + \cos\gamma\frac{d\delta\eta}{dz}\right), \qquad (F) \\
&\rho Z\left(\cos\alpha\frac{d\delta\zeta}{dx} + \cos\beta\frac{d\delta\zeta}{dy} + \cos\gamma\frac{d\delta\zeta}{dz}\right).
\end{aligned}
$$

Substituting for X its value from (E), we find the following expression for the moment of that force:

$$\begin{aligned}&\rho^2 \cos\alpha' \left(\cos\alpha \frac{d\delta\xi}{dx} + \cos\beta \frac{d\delta\xi}{dy} + \cos^2\gamma \frac{d\delta\xi}{dz}\right)\\ &\times \left\{ A \left(\cos\alpha \frac{d\xi}{dx} + \cos\beta \frac{d\xi}{dy} + \cos\gamma \frac{d\xi}{dz}\right)\right.\\ &\quad + B \left(\cos\alpha \frac{d\eta}{dx} + \cos\beta \frac{d\eta}{dy} + \cos\gamma \frac{d\eta}{dz}\right)\\ &\quad \left. + C \left(\cos\alpha \frac{d\zeta}{dx} + \cos\beta \frac{d\zeta}{dy} + \cos\gamma \frac{d\zeta}{dz}\right)\right\}.\end{aligned} \tag{G}$$

This expression denoting the moment of that part of the force acting on m, which results from the relative displacement of m', it is evident that the complete moment of the forces X, which act upon m, will be found by multiplying (G) by the element of the mass, and integrating through the entire sphere of molecular action. Let ϵ be the density at the point x', y', z', and a the radius of the sphere of molecular activity. Then the element of the mass will be

$$d\mu = \epsilon\rho^2 \sin\theta \, d\rho \, d\theta \, d\phi;$$

and the limits of integration with respect to ρ, θ, ϕ, will be 0 and a, 0 and π, 0 and 2π, respectively. If then we assume,

$$\begin{aligned}
A_{\alpha^2\alpha'} &= \iiint A\rho^2 \cos^2\alpha \cos\alpha' \, d\mu = \int_0^\pi\int_0^{2\pi}\int_0^a A\epsilon\rho^4 \cos\alpha' \sin^3\theta \cos^2\phi \, d\rho \, d\theta \, d\phi,\\
A_{\alpha\beta\alpha'} &= \iiint A\rho^2 \cos\alpha \cos\beta \cos\alpha' \, d\mu,\\
A_{\alpha\gamma\alpha'} &= \iiint A\rho^2 \cos\alpha \cos\gamma \cos\alpha' \, d\mu,\\
&\&c.;\\
B_{\alpha^2\alpha'} &= \iiint B\rho^2 \cos^2\alpha \cos\alpha' \, d\mu,\\
&\&c.;\\
C_{\alpha^2\alpha'} &= \iiint C\rho^2 \cos^2\alpha \cos\alpha' \, d\mu,\\
&\&c.;
\end{aligned}$$

we shall find for the complete moment of the forces X acting upon the particle m, the expression

$$L = A_{\alpha^2\alpha'} \frac{d\xi}{dx}\frac{d\delta\xi}{dx} + A_{\beta^2\alpha'} \frac{d\xi}{dy}\frac{d\delta\xi}{dy} + A_{\gamma^2\alpha'} \frac{d\xi}{dz}\frac{d\delta\xi}{dz}$$

$$+ A_{\beta\gamma\alpha'}\left(\frac{d\xi}{dy}\frac{d\delta\xi}{dz} + \frac{d\xi}{dz}\frac{d\delta\xi}{dy}\right) + A_{\alpha\gamma\alpha'}\left(\frac{d\xi}{dz}\frac{d\delta\xi}{dx} + \frac{d\xi}{dx}\frac{d\delta\xi}{dz}\right) + A_{\alpha\beta\alpha'}\left(\frac{d\xi}{dx}\frac{d\delta\xi}{dy} + \frac{d\xi}{dy}\frac{d\delta\xi}{dx}\right)$$

$$+ B_{\alpha^2\alpha'} \frac{d\eta}{dx}\frac{d\delta\xi}{dx} + B_{\beta^2\alpha'} \frac{d\eta}{dy}\frac{d\delta\xi}{dy} + B_{\gamma^2\alpha'} \frac{d\eta}{dz}\frac{d\delta\xi}{dz} \qquad \text{(H)}$$

$$+ B_{\beta\gamma\alpha'}\left(\frac{d\eta}{dy}\frac{d\delta\xi}{dz} + \frac{d\eta}{dz}.\frac{d\delta\xi}{dy}\right) + B_{\alpha\gamma\alpha'}\left(\frac{d\eta}{dz}\frac{d\delta\xi}{dx} + \frac{d\eta}{dx}\frac{d\delta\xi}{dz}\right) + B_{\alpha\beta\alpha'}\left(\frac{d\eta}{dx}\frac{d\delta\xi}{dy} + \frac{d\eta}{dy}\frac{d\delta\xi}{dx}\right)$$

$$+ C_{\alpha^2\alpha'} \frac{d\zeta}{dx}\frac{d\delta\xi}{dx} + C_{\beta^2\alpha'} \frac{d\zeta}{dy}\frac{d\delta\xi}{dy} + C_{\gamma^2\alpha'} \frac{d\zeta}{dz}\frac{d\delta\xi}{dz}$$

$$+ C_{\beta\gamma\alpha'}\left(\frac{d\zeta}{dy}\frac{d\delta\xi}{dz} + \frac{d\zeta}{dz}\frac{d\delta\xi}{dy}\right) + C_{\alpha\gamma\alpha'}\left(\frac{d\zeta}{dz}\frac{d\delta\xi}{dx} + \frac{d\zeta}{dx}\frac{d\delta\xi}{dz}\right) + C_{\alpha\beta\alpha'}\left(\frac{d\zeta}{dx}\frac{d\delta\xi}{dy} + \frac{d\zeta}{dy}\frac{d\delta\xi}{dx}\right).$$

Similar expressions are found for the moments of the forces Y and Z.

5. Let X', Y', Z' be the external forces necessary to keep the particle at rest. Then, the equation of virtual velocities being in general

$$\iiint (X'\delta\xi + Y'\delta\eta + Z'\delta\zeta)\, dm + \iiint (L + M + N)\, dx\,dy\,dz = 0;$$

if we substitute for L, M, N their values found as above, we shall have the equation

$$\begin{aligned} 0 = &\iiint (X'\delta\xi + Y'\delta\eta + Z'\delta\zeta)\, \epsilon\, dx\,dy\,dz \\ &+ \iiint \Big(P_1 \frac{d\delta\xi}{dx} + P_2 \frac{d\delta\xi}{dy} + P_3 \frac{d\delta\xi}{dz} \\ &\quad + Q_1 \frac{d\delta\eta}{dx} + Q_2 \frac{d\delta\eta}{dy} + Q_3 \frac{d\delta\eta}{dz} \\ &\quad + R_1 \frac{d\delta\zeta}{dx} + R_2 \frac{d\delta\zeta}{dy} + R_3 \frac{d\delta\zeta}{dz}\Big)\, dx\,dy\,dz. \end{aligned} \qquad \text{(I)}$$

Where ϵ is the density, and

$$\begin{aligned} P_1 = &A_{\alpha^2\alpha'} \frac{d\xi}{dx} + A_{\alpha\beta\alpha'} \frac{d\xi}{dy} + A_{\alpha\gamma\alpha'} \frac{d\xi}{dz} \\ &+ B_{\alpha^2\alpha'} \frac{d\eta}{dx} + B_{\alpha\beta\alpha'} \frac{d\eta}{dy} + B_{\alpha\gamma\alpha'} \frac{d\eta}{dz} \\ &+ C_{\alpha^2\alpha'} \frac{d\zeta}{dx} + C_{\alpha\beta\alpha'} \frac{d\zeta}{dy} + C_{\alpha\gamma\alpha'} \frac{d\zeta}{dz}; \end{aligned} \qquad \text{(K)}$$

$$P_2 = A_{\alpha\beta\alpha'} \frac{d\xi}{dx} + A_{\beta^2\alpha'} \frac{d\xi}{dy} + A_{\beta\gamma\alpha'} \frac{d\xi}{dz}$$
$$+ B_{\alpha\beta\alpha'} \frac{d\eta}{dx} + B_{\beta^2\alpha'} \frac{d\eta}{dy} + B_{\beta\gamma\alpha'} \frac{d\eta}{dz}$$
$$+ C_{\alpha\beta\alpha'} \frac{d\zeta}{dx} + C_{\beta^2\alpha'} \frac{d\zeta}{dy} + C_{\beta\gamma\alpha'} \frac{d\zeta}{dz};$$

$$P_3 = A_{\alpha\gamma\alpha'} \frac{d\xi}{dx} + A_{\beta\gamma\alpha'} \frac{d\xi}{dy} + A_{\gamma^2\alpha'} \frac{d\xi}{dz}$$
$$+ B_{\alpha\gamma\alpha'} \frac{d\eta}{dx} + B_{\beta\gamma\alpha'} \frac{d\eta}{dy} + B_{\gamma^2\alpha'} \frac{d\eta}{dz}$$
$$+ C_{\alpha\gamma\alpha'} \frac{d\zeta}{dx} + C_{\beta\gamma\alpha'} \frac{d\zeta}{dy} + C_{\gamma^2\alpha'} \frac{d\zeta}{dz};$$

the values of Q_1, Q_2, Q_3 being deduced from these expressions by changing, in the suffixed letters, α' into β'; and those of R_1, R_2, R_3, by changing α' into γ'.

Integrating by parts, and equating to zero the coefficients of $\delta\xi, \delta\eta, \delta\zeta$, under the triple sign of integration, we find the equations of equilibrium to be

$$\begin{aligned} \epsilon X' &= \frac{dP_1}{dx} + \frac{dP_2}{dy} + \frac{dP_3}{dz}, \\ \epsilon Y' &= \frac{dQ_1}{dx} + \frac{dQ_2}{dy} + \frac{dQ_3}{dz}, \\ \epsilon Z' &= \frac{dR_1}{dx} + \frac{dR_2}{dy} + \frac{dR_3}{dz}. \end{aligned} \tag{L}$$

The corresponding dynamical equations will be

$$\begin{aligned} \epsilon\left(X' - \frac{d^2\xi}{dt^2}\right) &= \frac{dP_1}{dx} + \frac{dP_2}{dy} + \frac{dP_3}{dz}, \\ \epsilon\left(Y' - \frac{d^2\eta}{dt^2}\right) &= \frac{dQ_1}{dx} + \frac{dQ_2}{dy} + \frac{dQ_3}{dz}, \\ \epsilon\left(Z' - \frac{d^2\zeta}{dt^2}\right) &= \frac{dR_1}{dx} + \frac{dR_2}{dy} + \frac{dR_3}{dz}. \end{aligned} \tag{M}$$

If now we suppose that no external forces act, and replace P_1, P_2, &c., by their values (K), we shall have the three general equations of small oscillations

in a body whose particles have been displaced from their original position of free equilibrium.

$$
\begin{aligned}
-\epsilon\frac{d^2\xi}{dt^2} = {} & A_{\alpha^2\alpha'}\frac{d^2\xi}{dx^2} + A_{\beta^2\alpha'}\frac{d^2\xi}{dy^2} + A_{\gamma^2\alpha'}\frac{d^2\xi}{dz^2} \\
& + B_{\alpha^2\alpha'}\frac{d^2\eta}{dx^2} + B_{\beta^2\alpha'}\frac{d^2\eta}{dy^2} + B_{\gamma^2\alpha'}\frac{d^2\eta}{dz^2} \\
& + C_{\alpha^2\alpha'}\frac{d^2\zeta}{dx^2} + C_{\beta^2\alpha'}\frac{d^2\zeta}{dy^2} + C_{\gamma^2\alpha'}\frac{d^2\zeta}{dz^2} \\
& + 2A_{\beta\gamma\alpha'}\frac{d^2\xi}{dydz} + 2A_{\alpha\gamma\alpha'}\frac{d^2\xi}{dxdz} + 2A_{\alpha\beta\alpha'}\frac{d^2\xi}{dxdy} \\
& + 2B_{\beta\gamma\alpha'}\frac{d^2\eta}{dydz} + 2B_{\alpha\gamma\alpha'}\frac{d^2\eta}{dxdz} + 2B_{\alpha\beta\alpha'}\frac{d^2\eta}{dxdy} \\
& + 2C_{\beta\gamma\alpha'}\frac{d^2\zeta}{dydz} + 2C_{\alpha\gamma\alpha'}\frac{d^2\zeta}{dxdz} + 2C_{\alpha\beta\alpha'}\frac{d^2\zeta}{dxdy} \\
& + \left(\frac{dA_{\alpha^2\alpha'}}{dx} + \frac{dA_{\alpha\beta\alpha'}}{dy} + \frac{dA_{\alpha\gamma\alpha'}}{dz}\right)\frac{d\xi}{dx} \\
& + \left(\frac{dA_{\alpha\beta\alpha'}}{dx} + \frac{dA_{\beta^2\alpha'}}{dy} + \frac{dA_{\beta\gamma\alpha'}}{dz}\right)\frac{d\xi}{dy} \\
& + \left(\frac{dA_{\alpha\gamma\alpha'}}{dx} + \frac{dA_{\beta\gamma\alpha'}}{dy} + \frac{dA_{\gamma^2\alpha'}}{dz}\right)\frac{d\xi}{dz} \\
& + \left(\frac{dB_{\alpha^2\alpha'}}{dx} + \frac{dB_{\alpha\beta\alpha'}}{dy} + \frac{dB_{\alpha\gamma\alpha'}}{dz}\right)\frac{d\eta}{dx} \\
& + \left(\frac{dB_{\alpha\beta\alpha'}}{dx} + \frac{dB_{\beta^2\alpha'}}{dy} + \frac{dB_{\beta\gamma\alpha'}}{dz}\right)\frac{d\eta}{dy} \\
& + \left(\frac{dB_{\alpha\gamma\alpha'}}{dx} + \frac{dB_{\beta\gamma\alpha'}}{dy} + \frac{dB_{\gamma^2\alpha'}}{dz}\right)\frac{d\eta}{dz} \\
& + \left(\frac{dC_{\alpha^2\alpha'}}{dx} + \frac{dC_{\alpha\beta\alpha'}}{dy} + \frac{dC_{\alpha\gamma\alpha'}}{dz}\right)\frac{d\zeta}{dx} \\
& + \left(\frac{dC_{\alpha\beta\alpha'}}{dx} + \frac{dC_{\beta^2\alpha'}}{dy} + \frac{dC_{\beta\gamma\alpha'}}{dz}\right)\frac{d\zeta}{dy}\,. \\
& + \left(\frac{dC_{\alpha\gamma\alpha'}}{dx} + \frac{dC_{\beta\gamma\alpha'}}{dy} + \frac{dC_{\gamma^2\alpha'}}{dz}\right)\frac{d\zeta}{dz};
\end{aligned}
\qquad \text{(N)}
$$

$$-\epsilon\frac{d^2\eta}{dt^2}=A_{a^2\beta'}\frac{d^2\xi}{dx^2}+\&c.;$$

$$-\epsilon\frac{d^2\zeta}{dt^2}=A_{a^2\gamma'}\frac{d^2\xi}{dx^2}+\&c.$$

Of these equations the second and third are deduced from the first by simply changing, in the suffixed letters, α' into β' and γ' respectively.

If the body be *homogeneous,* i. e. if all its points be absolutely similar, the quantities

$$A_{a^2a'},\ \&c.,\quad B_{a^2a'},\ \&c.,\quad C_{a^2a'},\ \&c.,$$

will be evidently constant. The terms involving

$$\frac{d\xi}{dx},\ \&c.,\quad \frac{d\eta}{dx},\ \&c.,\quad \frac{d\zeta}{dx},\ \&c.,$$

will, therefore, disappear, and the equations (N) will become homogeneous partial differential equations of the second order with constant coefficients.* The number of constants which these equations contain is the same as the number of terms in the right hand members, namely, eighteen for each equation. Hence it is evident, that the equations which represent the small oscillations of a homogeneous medium satisfying the hypothesis of independent action, contain, in general, fifty-four constants. This is the greatest number of constants which these equations could be made to have without a change of form.

6. The conditions to be fulfilled at the limits of integration are, of course, obtained from the terms which appear under a double sign of integration in the equation derived from (I). These terms will evidently give

$$\begin{aligned}&\iint(P_1\delta\xi+Q_1\delta\eta+R_1\delta\zeta)\,dy\,dz\\+&\iint(P_2\delta\xi+Q_2\delta\eta+R_2\delta\zeta)\,dz\,dx\\+&\iint(P_3\delta\xi+Q_3\delta\eta+R_3\delta\zeta)\,dx\,dy=0.\end{aligned}\qquad(\mathrm{N}')$$

Let p, q, r be the angles which the normal to the surface bounding the given medium makes with the axes, and let dS' be the element of this surface. Then, if equation (N′) be transformed in the usual way by making

* This conclusion does not hold for molecules situated at the surface of the body.—Vid. Art. 15.

$$dy\,dz = \cos p\,dS', \quad dz\,dx = \cos q\,dS', \quad dx\,dy = \cos r\,dS';$$

we shall have

$$\begin{aligned}&\iint (P_1 \cos p + P_2 \cos q + P_3 \cos r)\,\delta\xi\, dS' \\ +&\iint (Q_1 \cos p + Q_2 \cos q + Q_3 \cos r)\,\delta\eta\, dS' \\ +&\iint (R_1 \cos p + R_2 \cos q + R_3 \cos r)\,\delta\zeta\, dS' \\ +&\iint (X_1\,\delta\xi + Y_1\,\delta\eta + Z_1\,\delta\zeta)\,\epsilon_1\, dS' = 0;\end{aligned}$$

where X_1, Y_1, Z_1 are forces acting solely at the surface of the medium. The mode of treating this equation in the several cases which may occur having been fully given by Mr. HAUGHTON, I do not think it necessary to pursue this part of the subject further. On the most important of these cases, namely, the transmission of motion from one medium to another, vid. Art. 15.

7. We shall now proceed to integrate the equations (N), for the particular case of plane waves and rectilinear vibrations in a homogeneous body.

Assume,

$$\begin{aligned}\xi &= \cos l . f\,(ax + by + cz - vt),\\ \eta &= \cos m . f\,(ax + by + cz - vt),\\ \zeta &= \cos n . f\,(ax + by + cz - vt);\end{aligned}$$

where a, b, c are the cosines of the angles which the wave normal makes with the axes, and l, m, n are the angles made by the direction of vibration.

Substituting these values in (N), we find

$$\begin{aligned}-\epsilon v^2 \cos l &= \Pi_1 \cos l + \Phi_1 \cos m + \Psi_1 \cos n,\\ -\epsilon v^2 \cos m &= \Pi_2 \cos l + \Phi_2 \cos m + \Psi_2 \cos n,\\ -\epsilon v^2 \cos n &= \Pi_3 \cos l + \Phi_3 \cos m + \Psi_3 \cos n;\end{aligned} \tag{O}$$

where

$$\begin{aligned}\Pi_1 &= A_{\alpha^2\alpha'}\,a^2 + A_{\beta^2\alpha'}\,b^2 + A_{\gamma^2\alpha'}\,c^2 + 2A_{\beta\gamma\alpha'}\,bc + 2A_{\alpha\gamma\alpha'}\,ac + 2A_{\alpha\beta\alpha'}\,ab,\\ \Phi_1 &= B_{\alpha^2\alpha'}\,a^2 + B_{\beta^2\alpha'}\,b^2 + B_{\gamma^2\alpha'}\,c^2 + 2B_{\beta\gamma\alpha'}\,bc + 2B_{\alpha\gamma\alpha'}\,ac + 2B_{\alpha\beta\alpha'}\,ab,\\ \Psi_1 &= C_{\alpha^2\alpha'}\,a^2 + C_{\beta^2\alpha'}\,b^2 + C_{\gamma^2\alpha'}\,c^2 + 2C_{\beta\gamma\alpha'}\,bc + 2C_{\alpha\gamma\alpha'}\,ac + 2C_{\alpha\beta\alpha'}\,ab;\end{aligned} \tag{O'}$$

the values of Π_2, Φ_2, Ψ_2, Π_3, Φ_3, Ψ_3 being deduced from those of Π_1, Φ_1, Ψ_1, by changing, as before, α' into β' and γ'. Assuming

$$s = -\epsilon v^2,$$

and eliminating l, m, n between the equations (O), we find

$$(s-\Pi_1)(s-\Phi_2)(s-\Psi_3)-\Phi_3\Psi_2(s-\Pi_1)-\Psi_1\Pi_3(s-\Phi_2)-\Pi_2\Phi_1(s-\Psi_3)$$
$$-\Pi_2\Phi_3\Psi_1-\Pi_3\Phi_1\Psi_2=0. \qquad \text{(P)}$$

The value of s being determined by this equation, those of $\cos l$, $\cos m$, $\cos n$ are found at once from (O), combined with the condition

$$\cos^2 l+\cos^2 m+\cos^2 n=1.$$

Hence, the equation (P) being of the third degree, it appears that for each direction of wave plane there are in general three directions of molecular displacement; of these directions one is necessarily real, while the remaining two may be either both real or both imaginary. The vibration will not, however, be necessarily real, because its direction is so, as it is further necessary that the corresponding velocity of wave propagation should be real. Hence, as

$$s=-\epsilon v^2,$$

it is plain that at least one value of s must be negative. We infer, therefore, generally, that no body will be capable of transmitting a plane wave propagated by parallel rectilinear vibrations, unless equation (P) have at least one real negative root.

The surface of wave slowness, being the locus of a point upon the wave normal whose distance from the origin is inversely as the velocity of wave propagation, will be found by putting

$$s=-\frac{1}{r^2}$$

in the general equation (P). It is evidently a surface of the sixth order. In fact, if we put

$$P=A_{\alpha^2\alpha'}x^2+A_{\beta^2\alpha'}y^2+A_{\gamma^2\alpha'}z^2+2A_{\beta\gamma\alpha'}yz+2A_{\alpha\gamma\alpha'}xz+2A_{\alpha\beta\alpha'}xy,$$
$$Q=B_{\alpha^2\alpha'}x^2+\&c.,$$
$$R=C_{\alpha^2\alpha'}x^2+\&c.,$$

and denote by P', Q', R', P'', Q'', R'' the expressions derived from these by replacing α' by β' and γ' successively, we shall have, as the equation of the surface,

$$(P-1)(Q'-1)(R''-1)-R'Q''(P-1)-P''R(Q'-1)-QP'(R''-1) \\ +P'Q''R+P''QR'=0. \qquad \text{(Q)}$$

We shall next proceed to consider the two hypotheses which have been most frequently made by writers upon this subject, namely:—1. That the sum of the internal moments may be represented by the variation of a single function. 2. That the force which one molecule exerts upon another is a force of attraction or repulsion.

Hypothesis of the Existence of a single Function V, by whose Variation the Sum of the internal Moments of the Body may be represented.

8. This condition gives the equation

$$L+M+N=\delta V.$$

The three expressions (H), p. 187, must, therefore, when added together, give a complete variation. Now if we examine the value of L there given, we shall see that the first six terms, those, namely, which are multiplied by

$$A_{\alpha^2\alpha'},\ A_{\beta^2\alpha'},\ A_{\gamma^2\alpha'},\ A_{\beta\gamma\alpha'},\ A_{\alpha\gamma\alpha'},\ A_{\alpha\beta\alpha'};$$

form in themselves a complete variation, namely,

$$\delta\left\{\tfrac{1}{2}A_{\alpha^2\alpha'}\frac{d\xi^2}{dx^2}+\tfrac{1}{2}A_{\beta^2\alpha'}\frac{d\xi^2}{dy^2}+\tfrac{1}{2}A_{\gamma^2\alpha'}\frac{d\xi^2}{dz^2}+A_{\beta\gamma\alpha'}\frac{d\xi}{dy}\frac{d\xi}{dz} \right.$$
$$\left. +A_{\alpha\gamma\alpha'}\frac{d\xi}{dz}\frac{d\xi}{dx}+A_{\alpha\beta\alpha'}\frac{d\xi}{dx}\frac{d\xi}{dy}\right\}.$$

Similarly in the values of M and N the terms multiplied by

$$B_{\alpha^2\beta'},\ B_{\beta^2\beta'},\ \&c.,\qquad C_{\alpha^2\gamma'},\ C_{\beta^2\gamma'},\ \&c.,$$

respectively, form complete variations in themselves.

Let us now consider the term in the value of L

$$B_{\alpha^2\alpha'}\frac{d\eta}{dx}\frac{d\delta\xi}{dx}.$$

Corresponding to this term, we have in the value of M

$$A_{\alpha^2\beta'}\frac{d\xi}{dx}\frac{d\delta\eta}{dx};$$

and, therefore, since no such term can occur in N, we shall have in

$$L + M + N,$$

the group

$$B_{\alpha^2\alpha'} \frac{d\eta}{dx} \frac{d\delta\xi}{dx} + A_{\alpha^2\beta'} \frac{d\xi}{dx} \frac{d\delta\eta}{dx}. \qquad (Q')$$

Now if $L + M + N$ be a complete variation, it is plain that this expression must have been derived from a single term of the form

$$K \frac{d\eta}{dx} \frac{d\xi}{dx}.$$

Taking the variation of this term by the ordinary rule, and comparing it with (Q'), we have the condition

$$A_{\alpha^2\beta'} = B_{\alpha^2\alpha'}.$$

Proceeding in a similar way with the remaining terms, we find altogether eighteen equations of condition, namely,

$$\begin{array}{lll} A_{\alpha^2\beta'} = B_{\alpha^2\alpha'}, & A_{\beta^2\beta'} = B_{\beta^2\alpha'}, & A_{\gamma^2\beta'} = B_{\gamma^2\alpha'}, \\ B_{\alpha^2\gamma'} = C_{\alpha^2\beta'}, & B_{\beta^2\gamma'} = C_{\beta^2\beta'}, & B_{\gamma^2\gamma'} = C_{\gamma^2\beta'}, \\ C_{\alpha^2\alpha'} = A_{\alpha^2\gamma'}, & C_{\beta^2\alpha'} = A_{\beta^2\gamma'}, & C_{\gamma^2\alpha'} = A_{\gamma^2\gamma'}, \\ A_{\alpha\beta\beta'} = B_{\alpha\beta\alpha'}, & A_{\alpha\gamma\beta'} = B_{\alpha\gamma\alpha'}, & A_{\beta\gamma\beta'} = B_{\beta\gamma\alpha'}, \\ B_{\alpha\beta\gamma'} = C_{\alpha\beta\beta'}, & B_{\alpha\gamma\gamma'} = C_{\alpha\gamma\beta'}, & B_{\beta\gamma\gamma'} = C_{\beta\gamma\beta'}, \\ C_{\alpha\beta\alpha'} = A_{\alpha\beta\gamma'}, & C_{\alpha\gamma\alpha'} = A_{\alpha\gamma\gamma'}, & C_{\beta\gamma\alpha'} = A_{\beta\gamma\gamma'}. \end{array} \qquad (R)$$

These equations may be more briefly written as follows:

$$\begin{aligned} &\iiint (A \cos \beta' - B \cos \alpha') (\xi' - \xi) (\eta' - \eta) \, dm = 0, \\ &\iiint (C \cos \alpha' - A \cos \gamma') (\zeta' - \zeta) (\xi' - \xi) \, dm = 0, \\ &\iiint (B \cos \gamma' - C \cos \beta') (\eta' - \eta) (\zeta' - \zeta) \, dm = 0. \end{aligned} \qquad (S)$$

For if we substitute for

$$\xi' - \xi, \quad \eta' - \eta, \quad \zeta' - \zeta,$$

their values (C), and perform the integrations with regard to dm, the first of the foregoing equations may be written

$$(A_{\alpha^2\beta'} - B_{\alpha^2\alpha'})\frac{d\xi}{dx}\frac{d\eta}{dx} + (A_{\beta^2\beta'} - B_{\beta^2\alpha'})\frac{d\xi}{dy}\frac{d\eta}{dy} + (A_{\gamma^2\beta'} - B_{\gamma^2\alpha'})\frac{d\xi}{dz}\frac{d\eta}{dz}$$
$$+ (A_{\alpha\beta\beta'} - B_{\alpha\beta\alpha'})\left(\frac{d\xi}{dx}\frac{d\eta}{dy} + \frac{d\xi}{dy}\frac{d\eta}{dx}\right)$$
$$+ (A_{\alpha\gamma\beta'} - B_{\alpha\gamma\alpha'})\left(\frac{d\xi}{dx}\frac{d\eta}{dz} + \frac{d\xi}{dz}\frac{d\eta}{dx}\right)$$
$$+ (A_{\beta\gamma\beta'} - B_{\beta\gamma\alpha'})\left(\frac{d\xi}{dy}\frac{d\eta}{dz} + \frac{d\xi}{dz}\frac{d\eta}{dy}\right) = 0.$$

Since then these equations are supposed to hold for all possible displacements, we must have the six equations

$$A_{\alpha^2\beta'} = B_{\alpha^2\alpha'}, \quad A_{\beta^2\beta'} = B_{\beta^2\alpha'}, \quad A_{\gamma^2\beta'} = B_{\gamma^2\alpha'},$$
$$A_{\alpha\beta\beta'} = B_{\alpha\beta\alpha'}, \quad A_{\alpha\gamma\beta'} = B_{\alpha\gamma\gamma'}, \quad A_{\beta\gamma\beta'} = B_{\beta\gamma\alpha'}.$$

Six equations being furnished by each of the remaining equations (S), we shall have in all eighteen equations which are obviously identical with (R).

9. If the sum of the internal moments admit of being represented by the variation of a single function, the three directions of molecular displacement corresponding to a given wave plane will be at right angles to each other. This has been shown by Mr. HAUGHTON. We shall now proceed to prove the converse of this theorem, namely,

If the three directions of molecular displacement corresponding to the same wave plane be at right angles, and if this be true for every wave plane, the sum of the internal moments of the body may be represented by the variation of a single function.

We have seen that the directions of molecular displacement corresponding to a given wave plane are determined by the equations

$$-\epsilon v^2 \cos l = \Pi_1 \cos l + \Phi_1 \cos m + \Psi_1 \cos n,$$
$$-\epsilon v^2 \cos m = \Pi_2 \cos l + \Phi_2 \cos m + \Psi_2 \cos n,$$
$$-\epsilon v^2 \cos n = \Pi_3 \cos l + \Phi_3 \cos m + \Psi_3 \cos n.$$

Eliminating ϵv^2 between the first two of these equations, we have

$$\Pi_1 \cos l \cos m + \Phi_1 \cos^2 m + \Psi_1 \cos m \cos n$$
$$= \Pi_2 \cos^2 l + \Phi_2 \cos l \cos m + \Psi_2 \cos l \cos n.$$

Hence if $l_1, m_1, n_1,\ l_2, m_2, n_2,\ l_3, m_3, n_3$ be the three systems of values of l, m, n, we shall have

$$\begin{aligned}
&\Pi_1 \cos l_1 \cos m_1 + \Phi_1 \cos^2 m_1 + \Psi_1 \cos m_1 \cos n_1 \\
&= \Pi_2 \cos^2 l_1 + \Phi_2 \cos l_1 \cos m_1 + \Psi_2 \cos l_1 \cos n_1, \\
&\Pi_1 \cos l_2 \cos m_2 + \Phi_1 \cos^2 m_2 + \Psi_1 \cos m_2 \cos n_2 \\
&= \Pi_2 \cos^2 l_2 + \Phi_2 \cos l_2 \cos m_2 + \Psi_2 \cos l_2 \cos n_2, \\
&\Pi_1 \cos l_3 \cos m_3 + \Phi_1 \cos^2 m_3 + \Psi_1 \cos m_3 \cos n_3 \\
&= \Pi_2 \cos^2 l_3 + \Phi_2 \cos l_3 \cos m_3 + \Psi_2 \cos l_3 \cos n_3.
\end{aligned}$$

Adding these equations, and recollecting that, as the three directions of vibration are rectangular,

$$\begin{gathered}
\cos^2 l_1 + \cos^2 l_2 + \cos^2 l_3 = 1, \\
\cos^2 m_1 + \cos^2 m_2 + \cos^2 m_3 = 1, \\
\cos l_1 \cos m_1 + \cos l_2 \cos m_2 + \cos l_3 \cos m_3 = 0, \\
\cos l_1 \cos n_1 + \cos l_2 \cos n_2 + \cos l_3 \cos n_3 = 0, \\
\cos m_1 \cos n_1 + \cos m_2 \cos n_2 + \cos m_3 \cos n_3 = 0\,;
\end{gathered}$$

we have

$$\Pi_2 = \Phi_1,$$

and similarly

$$\Phi_3 = \Psi_2, \quad \Psi_1 = \Pi_3,$$

or, substituting for Π_2, &c., their values from (O′),

$$\begin{gathered}
(A_{\alpha^2\beta'} - B_{\alpha^2\alpha'})\, a^2 + (A_{\beta^2\beta'} - B_{\beta^2\alpha'})\, b^2 + (A_{\gamma^2\beta'} - B_{\gamma^2\alpha'})\, c^2 + 2(A_{\beta\gamma\beta'} - B_{\beta\gamma\alpha'})\, bc \\
+ 2(A_{\alpha\gamma\beta'} - B_{\alpha\gamma\alpha'})\, ac + 2(A_{\alpha\beta\beta'} - B_{\alpha\beta\alpha'})\, ab = 0,
\end{gathered}$$

$$\begin{gathered}
(B_{\alpha^2\gamma'} - C_{\alpha^2\beta'})\, a^2 + (B_{\beta^2\gamma'} - C_{\beta^2\beta'})\, b^2 + (B_{\gamma^2\gamma'} - C_{\gamma^2\beta'})\, c^2 \\
+ 2(B_{\beta\gamma\gamma'} - C_{\beta\gamma\beta'})\, bc + 2(B_{\alpha\gamma\gamma'} - C_{\alpha\gamma\beta'})\, ac + 2(B_{\alpha\beta\gamma'} - C_{\alpha\beta\beta'}) = 0,
\end{gathered}$$

$$\begin{gathered}
(C_{\alpha^2\alpha'} - A_{\alpha^2\gamma'})\, a^2 + (C_{\beta^2\alpha'} - A_{\beta^2\gamma'})\, b^2 + (C_{\gamma^2\alpha'} - A_{\gamma^2\gamma'})\, c^2 \\
+ 2(C_{\beta\gamma\alpha'} - A_{\beta\gamma\gamma'})\, bc + 2(C_{\alpha\gamma\alpha'} - A_{\alpha\gamma\gamma'})\, ac + 2(C_{\alpha\beta\alpha'} - A_{\alpha\beta\gamma'})\, ab = 0.
\end{gathered}$$

If these equations hold for all directions of wave plane, it is easily seen that the coefficients of

$$a^2,\ b^2,\ c^2, \quad ab,\ ac,\ bc,$$

must vanish of themselves. This condition will give eighteen equations which

are evidently identical with the system of equations (R). The theorem, as stated above, is, therefore, true.

10. The total number of constants in $L+M+N$ being fifty-four, it is evident that the number of *distinct* constants contained in δV, and, therefore, in V, will be

$$54 - 18 = 36.$$

Now V, which is, as we have seen, a homogeneous quadratic function of the nine quantities

$$\frac{d\xi}{dx}, \ \frac{d\xi}{dy}, \ \frac{d\xi}{dz}, \ \frac{d\eta}{dx}, \ \frac{d\eta}{dy}, \ \frac{d\eta}{dz}, \ \frac{d\zeta}{dx}, \ \frac{d\zeta}{dy}, \ \frac{d\zeta}{dz},$$

will contain in general forty-five terms, and therefore, if it be subjected to no restriction, forty-five distinct constants. The function at which we have arrived is not, therefore, in its most general form. In fact, if we examine the composition of the terms in the value of L (H), we see that the quantities

$$\frac{d\eta}{dy}\,\frac{d\delta\xi}{dz}, \quad \frac{d\eta}{dz}\,\frac{d\delta\xi}{dy},$$

have the same coefficient, namely $B_{\beta\gamma\alpha'}$. Similarly, in the value of M we should have two terms

$$\frac{d\xi}{dz}\,\frac{d\delta\eta}{dy}, \quad \frac{d\xi}{dy}\,\frac{d\delta\eta}{dz},$$

with the common coefficient $A_{\beta\gamma\beta'}$. These coefficients being, by the equations (R), identical, the four terms enumerated above may be written

$$B_{\beta\gamma\alpha'}\left(\frac{d\eta}{dy}\,\frac{d\delta\xi}{dz} + \frac{d\xi}{dz}\,\frac{d\delta\eta}{dy} + \frac{d\eta}{dz}\,\frac{d\delta\xi}{dy} + \frac{d\xi}{dy}\,\frac{d\delta\eta}{dz}\right),$$

or,

$$B_{\beta\gamma\alpha'}\,\delta\left(\frac{d\eta}{dy}\,\frac{d\xi}{dz} + \frac{d\eta}{dz}\,\frac{d\xi}{dy}\right).$$

Hence it is evident, that the terms in V containing

$$\frac{d\eta}{dy}\,\frac{d\xi}{dz}, \quad \frac{d\eta}{dz}\,\frac{d\xi}{dy},$$

will be of the form

$$I\left(\frac{d\eta}{dy}\frac{d\xi}{dz}+\frac{d\eta}{dz}\frac{d\xi}{dy}\right).$$

Similar conclusions will be obtained for all terms of this form. These terms are distinguished by the technical rule, that, in the products which they severally contain, the same letter does not occur twice. Thus the conclusion at which we have arrived does not apply to terms of the form

$$G\frac{d\xi}{dx}\frac{d\xi}{dy},$$

where the same letter ξ occurs twice; nor to terms of the form

$$H\frac{d\xi}{dx}\frac{d\eta}{dx},$$

in which the letter x occurs twice. The preceding discussion gives us, therefore, the following general theorem:

If the constitution of a body, whose particles act independently, and whose original position is one of free equilibrium, be such, that the sum of the internal moments may be represented by the variation of a single function, this function must be of the form

$$V=\Sigma\left(F\frac{d\xi^2}{dx^2}\right)+\Sigma\left(G\frac{d\xi}{dx}\frac{d\xi}{dy}\right)+\Sigma\left(H\frac{d\xi}{dx}\frac{d\eta}{dx}\right)$$
$$+\Sigma\left\{I\left(\frac{d\eta}{dy}\frac{d\xi}{dz}+\frac{d\eta}{dz}\frac{d\xi}{dy}\right)\right\}. \qquad (T)$$

Each of the sums denoted by Σ will contain nine terms, thus giving thirty-six for the total number of distinct constants in V.

11. Previously to proceeding further, it may be well to compare this result with the investigations of Professor MAC CULLAGH and Mr. GREEN, in the undulatory theory of light. Both these writers assume the original state of the supposed luminous ether to be one of free equilibrium.* Both suppose also, that the sum of the internal moments may be represented by the variation of a

* Mr. GREEN has also investigated the problem under the supposition that the original position is not one of free equilibrium. The remarks in the text are, of course, only meant to apply to the first supposition.

single function. This function, in the system of Professor MAC CULLAGH, is in its simplest form given by the equation

$$-2V = a^2\left(\frac{d\zeta}{dy} - \frac{d\eta}{dz}\right)^2 + b^2\left(\frac{d\xi}{dz} - \frac{d\zeta}{dx}\right)^2 + c^2\left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)^2.$$

The function used by Mr. GREEN is given by the equation

$$\begin{aligned} -2V &= \mu\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right)^2 \\ &+ L\left\{\left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right)^2 - 4\frac{d\eta}{dy}\frac{d\zeta}{dz}\right\} \\ &+ M\left\{\left(\frac{d\xi}{dz} + \frac{d\zeta}{dx}\right)^2 - 4\frac{d\xi}{dx}\frac{d\zeta}{dz}\right\} \\ &+ N\left\{\left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right)^2 - 4\frac{d\xi}{dx}\frac{d\eta}{dy}\right\}. \end{aligned}$$

Comparing these successively with the general form (T), p. 198, we see that the function used by Professor MAC CULLAGH cannot by any supposition be identified with (T), inasmuch as it contains the products

$$\frac{d\zeta}{dy}\frac{d\eta}{dz}, \quad \frac{d\xi}{dz}\frac{d\zeta}{dx}, \quad \frac{d\eta}{dx}\frac{d\xi}{dy},$$

without the corresponding products

$$\frac{d\zeta}{dz}\frac{d\eta}{dy}, \quad \frac{d\xi}{dx}\frac{d\zeta}{dz}, \quad \frac{d\eta}{dy}\frac{d\xi}{dx}.$$

To identify the function used by Mr. GREEN with (T), we should have, as is easily seen,

$$L = M = N = \tfrac{1}{3}\mu.$$

These conditions would render the body uncrystalline, and therefore incapable of being generally identical with the luminous ether.

Hence we infer, that if the supposed luminous ether be a medium such as either of these writers assume it to be, the mutual action of its particles cannot be independent. In other words, we must suppose that in such a medium the capacity which each particle possesses of exerting force on any other particle, is *modified* by the action of the surrounding particles.

Bodies composed of attracting and repelling Molecules.

12. Two conditions may be supposed to be included in the supposition, that the molecular force is a force of attraction or repulsion, namely:

1. That the *direction* of the force is in the line joining the molecules. 2. That the *intensity* of this force, for each pair of molecules, is represented by a function of the distance. Retaining the former of these conditions, we may replace the second by the hypothesis made in the foregoing section, namely, that the sum of the internal moments may be represented by the variation of a single function. For as the effect of the force is in this case to change the distance between two molecules, if this force be represented by F, and the distance between the particles by ρ, the moment of the force F will be

$$F\delta\rho,$$

or dividing the force F as before into F_0 and F_1, and putting $\rho+\rho'$ for the distance as changed by displacement,

$$\text{effective moment} = F_1\delta\rho',$$

and therefore the complete moment is expressed by

$$\iiint F_1\delta\rho' dm,$$

which cannot be a complete variation, unless

$$F_1 = f(\rho').$$

Hence the proposition is evident. Instead, therefore, of defining the body to be one composed of attracting or repelling molecules, we shall define it to be "a body in which the molecular force acts in the direction of the line joining the molecules, and in which the sum of the internal moments is represented by the variation of a single function." We shall consider successively the simplifications which these two suppositions introduce into the general equations.

The first hypothesis, that, namely, which regards the *direction* of the molecular force, is mathematically represented by making

$$\alpha' = \alpha, \quad \beta' = \beta, \quad \gamma' = \gamma.$$

If these conditions be introduced into the general equations (N), it is easily seen that the number of distinct constants will be reduced to thirty, *sc.*,

$$
\begin{aligned}
&A_{\alpha^3},\ A_{\beta^3},\ A_{\gamma^3},\ A_{\beta^2\gamma},\ A_{\gamma^2\beta},\ A_{\gamma^2\alpha},\ A_{\alpha^2\gamma},\ A_{\alpha^2\beta},\ A_{\beta^2\alpha},\ A_{\alpha\beta\gamma},\\
&B_{\alpha^3},\ \&c.\\
&C_{\alpha^3},\ \&c.
\end{aligned}
$$

The equations so reduced will refer to a system of attracting or repelling molecules, in the more enlarged sense of the term attraction or repulsion, the force being defined solely by its direction, without any hypothesis as to its intensity. Using the words in this sense, we may state the conclusion at which we have arrived as follows:

The equations of equilibrium or motion in a system of attracting or repelling molecules, will in general contain thirty distinct constants.

13. We shall next proceed to consider what further simplification is introduced into these equations by the supposition that the sum of the internal moments may be represented by the variation of a single function. Making

$$\alpha' = \alpha, \quad \beta' = \beta, \quad \gamma' = \gamma,$$

in the equations of condition (R), we shall find that their number will be reduced to fifteen, the nine equations

$$
\begin{aligned}
A_{\gamma^2\beta'} &= B_{\gamma^2\alpha'}, & C_{\beta\gamma\alpha'} &= A_{\beta\gamma\gamma'}, & B_{\alpha\gamma\gamma'} &= C_{\alpha\gamma\beta'},\\
A_{\beta\gamma\beta'} &= B_{\beta\gamma\alpha'}, & C_{\beta^2\alpha'} &= A_{\beta^2\gamma'}, & B_{\alpha\beta\gamma'} &= C_{\alpha\beta\beta'},\\
A_{\alpha\gamma\beta'} &= B_{\alpha\gamma\alpha'}, & C_{\alpha\beta\alpha'} &= A_{\alpha\beta\gamma'}, & B_{\alpha^2\gamma'} &= C_{\alpha^2\beta'},
\end{aligned}
$$

being obviously equivalent to but six. Hence we infer that

The equations of equilibrium or motion of a body in which the molecular force acts in the line joining the molecules, and is represented by a function of the distance, will contain fifteen distinct constants.

This agrees with the result obtained by Mr. HAUGHTON, to whose Memoir the reader is referred for the further discussion of this case.*

* Vid. note at the conclusion of this Memoir.

II.—HYPOTHESIS OF MODIFIED ACTION.

14. Let m, m' be two molecules of the medium under consideration, m being that whose equilibrium or motion is required. Then if, as before, we suppose the force which m' exerts upon m to be composed of two parts, one depending upon the relative displacement of these two particles, and the other existing previously to the displacement of either, we shall still have, as in p. 183,

$$F = F_0 + A(\xi' - \xi) + B(\eta' - \eta) + C(\zeta' - \zeta).$$

Now it is easily seen that the difference between this case and the preceding will show itself in the nature of the quantity F_0. In the former case, in which the action of m' is independent of the other particles of the medium, F_0 must be of the form

$$f(x, y, z, \quad \rho, \theta, \phi),$$

and may, as we have seen, be neglected in the case of a body whose original position is one of free equilibrium. But in the present case, in which it is supposed that the displacement of the other particles has itself the power of developing a force between m' and m, the form of F_0 is completely changed. Our first object, then, must be to determine the new form to be assigned to this quantity.

Let m'' be a third molecule of the given medium; ξ'', η'', ζ'', its displacements; and ρ_1, θ_1, ϕ_1, or $\rho_1, \alpha_1, \beta_1, \gamma_1$, its polar co-ordinates with regard to m. Then it will appear, by reasoning similar to that of p. 183, that the mathematical expression for its effect in developing a force between m and m' will be

$$f(x, y, z, \quad \rho, \theta, \phi, \quad \rho_1, \theta_1, \phi_1, \ \xi'' - \xi, \ \eta'' - \eta, \ \zeta'' - \zeta, \ \xi'' - \xi', \ \eta' - \eta', \ \zeta'' - \zeta');$$

or, as it may be otherwise written,

$$f(x, y, z, \quad \rho, \theta, \phi, \quad \rho_1, \theta_1, \phi_1, \ \xi'' - \xi, \ \eta'' - \eta, \ \zeta'' - \zeta, \ \xi' - \xi, \ \eta' - \eta, \ \zeta' - \zeta).$$

Treating this expression as in p. 184, it becomes

$$f_0 + a\rho\left(\cos\alpha\frac{d\xi}{dx} + \cos\beta\frac{d\xi}{dy} + \cos\gamma\frac{d\xi}{dz}\right)$$
$$+ b\rho\left(\cos\alpha\frac{d\eta}{dx} + \cos\beta\frac{d\eta}{dy} + \cos\gamma\frac{d\eta}{dz}\right)$$

$$+ c\rho \left(\cos\alpha \frac{d\zeta}{dx} + \cos\beta \frac{d\zeta}{dy} + \cos\gamma \frac{d\zeta}{dz} \right)$$

$$+ a_1 \rho_1 \left(\cos\alpha_1 \frac{d\xi}{dx} + \cos\beta_1 \frac{d\xi}{dy} + \cos\gamma_1 \frac{d\xi}{dz} \right)$$

$$+ b_1 \rho_1 \left(\cos\alpha_1 \frac{d\eta}{dx} + \cos\beta_1 \frac{d\eta}{dy} + \cos\gamma_1 \frac{d\eta}{dz} \right)$$

$$+ c_1 \rho_1 \left(\cos\alpha_1 \frac{d\zeta}{dx} + \cos\beta_1 \frac{d\zeta}{dy} + \cos\gamma_1 \frac{d\zeta}{dz} \right),$$

where f_0 is the force which is independent of the displacements of any one of the three particles. First let it be supposed that f_0 is independent of the displacements of any other particle.* Then, as the foregoing expression represents that part of the modifying force which results from the relative displacement of m'', it seems that the most general supposition which we can make as to the aggregate effect of all the particles is, that it is estimated by multiplying this expression by some function of the polar co-ordinates of m'', as also by the element of the mass, and integrating through the whole sphere of molecular activity. It is easily seen, that the result of this process will be an expression of the form

$$E_0 + A_1 \frac{d\xi}{dx} + A_2 \frac{d\xi}{dy} + A_3 \frac{d\xi}{dz}$$

$$+ B_1 \frac{d\eta}{dx} + B_2 \frac{d\eta}{dy} + B_3 \frac{d\eta}{dz}$$

$$+ C_1 \frac{d\zeta}{dx} + C_2 \frac{d\zeta}{dy} + C_3 \frac{d\zeta}{dz};$$

where E_0, A_1, B_1, &c., are definite integrals depending upon the constitution of the medium, being in general of the form

$$f(x, y, z, \quad \rho, \theta, \phi).$$

Hence the general value of F ((D), p. 185) will become

* It is easily seen, that this supposition does not limit the generality of the result.

$$E_0 + (A\rho \cos \alpha + A_1)\frac{d\xi}{dx} + (A\rho \cos \beta + A_2)\frac{d\xi}{dy} + (A\rho \cos \gamma + A_3)\frac{d\xi}{dz}$$
$$+ (B\rho \cos \alpha + B_1)\frac{d\eta}{dx} + (B\rho \cos \beta + B_2)\frac{d\eta}{dy} + (B\rho \cos \gamma + B_3)\frac{d\eta}{dz}$$
$$+ (C\rho \cos \alpha + C_1)\frac{d\zeta}{dx} + (C\rho \cos \beta + C_2)\frac{d\zeta}{dy} + (C\rho \cos \gamma + C_3)\frac{d\zeta}{dz}.$$

Resolving this force as before along the three axes, and proceeding as in p. 186, we find an expression for L similar to (H). There is, however, one important difference. In the value of L, which is derived from the principle of independent action, the quantities

$$\frac{d\xi}{dz}\frac{d\delta\xi}{dy}, \qquad \frac{d\xi}{dy}\frac{d\delta\xi}{dz},$$

have the same coefficient; and the same is true of the quantities

$$\frac{d\xi}{dx}\frac{d\delta\xi}{dz}, \qquad \frac{d\xi}{dz}\frac{d\delta\xi}{dx},$$
$$\frac{d\xi}{dy}\frac{d\delta\xi}{dx}, \qquad \frac{d\xi}{dx}\frac{d\delta\xi}{dy},$$
$$\frac{d\eta}{dz}\frac{d\delta\xi}{dy}, \qquad \frac{d\eta}{dy}\frac{d\delta\xi}{dz},$$
$$\frac{d\eta}{dx}\frac{d\delta\xi}{dz}, \qquad \frac{d\eta}{dz}\frac{d\delta\xi}{dx},$$
$$\frac{d\eta}{dy}\frac{d\delta\xi}{dx}, \qquad \frac{d\eta}{dx}\frac{d\delta\xi}{dy},$$
$$\frac{d\zeta}{dz}\frac{d\delta\xi}{dy}, \qquad \frac{d\zeta}{dy}\frac{d\delta\xi}{dz},$$
$$\frac{d\zeta}{dx}\frac{d\delta\xi}{dz}, \qquad \frac{d\zeta}{dz}\frac{d\delta\xi}{dx},$$
$$\frac{d\zeta}{dy}\frac{d\delta\xi}{dx}, \qquad \frac{d\zeta}{dx}\frac{d\delta\xi}{dy}.$$

Now it is easy to see that in general this restriction has no effect in limiting the generality of the equations of motion of a homogeneous body. For whether the coefficients be equal or not, each of the foregoing pairs of quan-

tities will furnish but *one* term to the equations of motion. Thus, if the quantities

$$\frac{d\xi}{dz}\,\frac{d\delta\xi}{dy}, \qquad \frac{d\xi}{dy}\,\frac{d\delta\xi}{dz},$$

enter into L in the form

$$A\,\frac{d\xi}{dz}\,\frac{d\delta\xi}{dy} + B\,\frac{d\xi}{dy}\,\frac{d\delta\xi}{dz},$$

the equations of motion will derive from them the single term

$$(A+B)\,\frac{d^2\xi}{dydz}.$$

It is evident, therefore, that the supposition

$$A = B$$

will not restrict in any way the generality of these equations. We have seen accordingly, that the principle of independent action gives to these equations the greatest number of independent constants which they can have, without a change of form. But the restriction may show itself in other ways. Thus, when we assume that the sum of the internal moments may be represented by the variation of a single function V, we find that in order to reconcile this supposition with the principle of independent action, it is necessary to assume further, that the coefficients of V are connected by nine equations of condition, and that, therefore, that principle does not admit of the existence of a function V in its most general form. This restriction has evidently been removed by supposing the state of each molecule to be modified by the action of the surrounding molecules. For, as we have just seen, this supposition enables us to obtain values for L, M, N in which the coefficients are completely independent of each other; and, with regard to the particular case of physical optics, we infer, as before, that if a luminous ether exist, whose constitution agrees with either of the hypotheses advanced by Professor MAC CULLAGH and Mr. GREEN respectively, each of the particles of that medium must be supposed to be capable of *modifying* the force exerted by any other particle within its sphere of action.

It is unnecessary to pursue the consequences of this principle further; for, as we have already seen, all the varieties of the general equations of motion, to the consideration of which the present Memoir is specially devoted, may be obtained from the more limited principle of independent action.

15. It is usual with writers upon the subject which has been here discussed, to consider the problem of the transmission of undulations from one body to another with which it is in mathematical contact. This problem, which, by an extension of the phraseology of optics, has been denominated the problem of refraction, has been investigated with special reference to a luminous ether, by Professor MAC CULLAGH, CAUCHY, GREEN, and others; and has been discussed by Mr. HAUGHTON for the case of solid bodies in general. But all these investigations appear to me to be liable to an objection to which I am unable to conceive any satisfactory answer. The nature of this objection, which has deterred me from following in this particular the steps of the writers in question, I shall now proceed to state.

On referring to p. 189 it will be seen, that the *form* of the general equations of motion, upon which the whole theory of undulation is based, depends upon the fact, that the coefficients are *constant* quantities, a fact which is, as we have seen, a result of the *homogeneity* of the medium; and the conclusions of p. 192 are evidently true, so long as the molecule under consideration is situated at a finite distance from the bounding surface of the medium. The functions to be integrated retaining the same form, and the limits of integration being the same, it is evident that the definite integrals will have the same value for every point.

Let us now consider the case of two media in contact. For the sake of simplicity, let the common surface of contact be an indefinite plane, which we shall take for the plane of xy. Let a, a' be the radii of molecular activity for the two media, and suppose that the molecule under consideration is situated at a distance from the plane of xy less than the greater of these. If now two*

* Instead of two spheres we may (as is easily seen) substitute a single sphere described with a radius not less than the greater radius of molecular activity. This substitution does not, however, in any way affect the reasoning in the text. The single definite integral

$$\iiint A \cos^2 \alpha \cos \alpha' dm$$

will still be replaced by

$$\iiint A \cos^2 \alpha \cos \alpha' dm + \iiint A_1 \cos^2 \alpha_1 \cos \alpha_1' dm;$$

the first being extended through the upper segment of the sphere, and the second through the lower segment. The value of the sum of these two quantities will evidently depend upon the distance of the point from the surface of separation.

spheres be described, with this molecule as their common centre, and with the radii a, a' respectively, each of the definite integrals of p. 186 will consist of two parts, the first being extended through all that portion of the first sphere which lies within the first medium; and the second through all that portion of the second sphere which lies in the second medium. Thus, instead of the definite integral

$$\iiint A \cos^2 \alpha \cos \alpha' \, dm,$$

taken through the entire of a sphere whose radius is a, we should have

$$\iiint A \cos^2 \alpha \cos \alpha' \, dm + \iiint A_1 \cos^2 \alpha_1 \cos \alpha_1' \, dm,$$

the limits of integration in each of these being determined as above stated. The limits of integration, and therefore the value of each of these integrals, depending upon the distance of the molecule from the plane of separation, it is evident that the coefficients in the general equations (N) will be functions of z, whose form will depend upon the constitutions of the two media, and will be, therefore, in general, unknown. The form of the equations of motion will therefore be completely altered, not only by the change of constant into variable coefficients, but by the introduction of terms of the first order,

$$\frac{d\xi}{dx}, \ \frac{d\xi}{dy}, \ \&c. \ \frac{d\eta}{dx}, \ \&c. \ \frac{d\zeta}{dx}, \ \&c.$$

The integral which represents wave motion will, therefore, be no longer applicable, nor will it be possible to give any integral of these equations without forming a number of additional hypotheses as to the constitution of the medium.

From these mathematical considerations, the following physical conclusions appear to be legitimately inferred:

(1.) That in the case of a single medium of limited extent, the molecules which are situated at a distance from the bounding surface less than the radius of molecular activity, move according to a law altogether different from that which regulates the motion of the particles in the interior.

(2.) That it is impossible to assign this law without forming one or more hypotheses as to the nature of the medium.

(3.) That if a plane wave pass through a homogeneous medium, it will not in general reach the surface; that is to say, the motion of the particles in and im-

mediately adjoining the surface will not be a wave motion composed of rectilinear vibrations.

(4.) That if two media be in contact, there will be a stratum of particles extending on each side of the surface of separation to a distance equal to the greatest radius of molecular activity; and that the motion of the particles in this stratum is altogether different from that of the particles in the interior of either medium.

(5.) That, therefore, two media which are thus in contact, may be each perfectly capable of transmitting plane waves through them in all directions, and yet incapable of transmitting such a motion from one to the other; and that even in the case of reflexion, in which the motion is transmitted back again through the same medium, the vibrations may cease to be rectilinear. The phenomenon of total reflexion affords an instance of this.

Now in the investigation of the problem of refraction, it is supposed that the integral which represents plane waves is applicable to the motion of the molecules which are actually situated in the surface of separation, a supposition which the foregoing considerations prove to be generally untrue. Nor does the truth of this conclusion depend upon the method employed in the previous discussion. On whatever principle we investigate the motion of the particles of a medium, it is easily seen, that for all points situated in the stratum described above, the medium cannot be considered homogeneous, inasmuch as the force to which each molecule is subject varies with its distance from the surface of separation. Within this stratum, therefore, the molecules must be considered as forming a *heterogeneous* medium, whose constitution varies rapidly according to some unknown law. It is difficult to see what modification is thus introduced into the discussion of the problem of refraction, in which the two media are supposed to be homogeneous. But it appears to me, that the supposition of plane wave motion extending to the mathematical limits of a medium is in general untenable. Nor shall we remove the difficulty in question by the supposition, that the molecules of one medium are incapable of influencing those of another. The only effect of such a supposition, which is, besides, wholly gratuitous, would be the substitution of one integral such as

$$\iiint A \cos^2 \alpha \cos \alpha' \, dm,$$

for the sum of two,

$$\iiint A \cos^2 a \cos a' dm + \iiint A_1 \cos^2 a_1 \cos a_1' dm_1.$$

But as the limits of integration are still variable, the form of the general equations of motion will still be that described in p. 207. These equations do not, as we have seen, admit of an integral representing plane wave motion. It is easily shown that the difficulty here alluded to does not affect that part of the theory of light or sound in which the *direction* of the reflected or refracted ray is derived from the consideration of wave motion.

16. Before concluding the present Memoir, I think it necessary to say a few words on the applicability of the integral calculus to problems like the present, or more generally to any problems in which bodies are considered, not as continuous masses, but as assemblages of distinct molecules.

I may remark, in the first place, that the method and results of the present Paper would be in no wise affected by the rejection of the molecular hypothesis; all that is essential to the validity of the method here given being attained by defining a molecule to be *a particle so small, that the motion of the system may be fully represented by the motions of all these particles considered as units;* and without such a supposition no equations of motion of a continuous body appear to have a perfectly definite meaning.

But as the constitution of the bodies which we find in nature appears to favour the supposition of separate molecules, rather than that of perfect continuity, it becomes an important question to determine how far the methods of the integral calculus are applicable to such cases. This is the more necessary, as M. POISSON denies the applicability of these methods to any problems connected with molecular force; and, more generally, to any problems in which the force varies with extreme rapidity within the limits of integration:—" Au reste la formule d'EULER qui sert à transformer les sommes en intégrales, contient une série ordonnée suivant les puissances de la différence finie de la variable, qui n'est pas toujours convergente, quoique cette différence soit supposée très petite. L'exception a lieu surtout dans le cas des fonctions comme $f(r)$ qui varient très rapidement."*

It is quite true, that the methods of the integral calculus are in strictness applicable only to continuous masses, and that it is in such cases only that the

* Mem. de l'Inst. tom. viii. p. 399.

results which it furnishes are mathematically accurate. When the mass ceases to be continuous, these results become approximate, and would of course be valueless, unless we had some means of testing the degree of approximation attained. This we shall now proceed to consider.

Let m be the mass of any one of the separate molecules of which the body is composed, and let x, y, z be its co-ordinates. Let $mf(x, y, z)$ be the mathematical expression of some quality or power belonging to this molecule, of such a nature, that the corresponding quality of the entire body is mathematically expressed by the *sum* of the expressions which refer to the several molecules. Let also m', x', y', z', m'', x'', y'', z'', &c., be the masses and co-ordinates of the other molecules. Then, if we assume

$$u = f(x, y, z), \quad u' = f(x', y', z'), \text{ \&c.},$$

the accurate expression sought for will be

$$mu + m'u' + m''u'' + \text{\&c.} = \Sigma mu.$$

Now let dv be the element of the volume geometrically considered, and ϵ the mean density of the matter which occupies it, so that its weight may be represented by

$$g\epsilon dv.$$

Then the approximate equation furnished by the integral calculus will be

$$\Sigma mu = \int u\epsilon dv.$$

In order to estimate the amount of the error which is involved in the use of the integral sign instead of the symbol of finite summation, we shall consider successively the several suppositions which are made in the interchange of these symbols, and the amount of the error introduced by each.

The object of this investigation being to determine, not the actual magnitude of the error, but merely its order, it is in the first place necessary to establish a notation to represent the respective orders of the several small quantities with which we are concerned.

Let e be an indefinitely small quantity which we take as the standard. Let the distance ω, between two consecutive molecules, be of the order i, or in other words let

$$\omega = ke^i;$$

where k is a finite magnitude. Let i' denote the degree of rapidity with which the function u varies, i. e. let it be supposed that u receives a finite increment in passing from one to another of two molecules, whose mutual distance ω' is given by the equation

$$\omega' = k'e^{i'}.$$

Suppose now the entire geometrical space which is occupied by the system of molecules, including also the small intervals or *pores* which separate them, to be divided into an indefinite number of equal portions, v, the linear dimension of each of which is a quantity of the order i''. We shall then have

$$v = k''^3 e^{3i''}.$$

Let $\Sigma_1 mu$ denote a finite summation extended to all the molecules contained in the first of these elements, $\Sigma_2 mu$ a similar summation for the molecules of the second element, $\Sigma_3 mu$ for the third, &c. Then

$$\Sigma mu = \Sigma_1 mu + \Sigma_2 mu + \Sigma_3 mu + \&c.$$

This equation is evidently exact.

Now let the following suppositions be made:

(1.) That u retains the same value for every molecule within the element v.

(2.) That the coefficient ϵ, which represents the mean density, is independent of the magnitude of the element.

These suppositions will give the following equations:

$$\Sigma_1 mu = u_1 \Sigma_1 m = u_1 \epsilon_1 v_1,$$

$$\Sigma_2 mu = u_2 \Sigma_2 m = u_2 \epsilon_2 v_2,$$

$$\&c. \quad \&c.;$$

and, therefore,

$$\Sigma mu = u_1 \epsilon_1 v_1 + u_2 \epsilon_2 v_2 + \&c. = \Sigma u \epsilon v.$$

Finally, instead of the symbol of finite summation Σ, let us substitute the symbol of integration $\int$, and we shall have

$$\Sigma mu = \int u \epsilon dv = \int u d\mu.$$

Let us now consider the order of the error introduced by each of these suppositions.

(1.) The supposition that u remains the same for all molecules situated within the element v, will introduce an error whose order is the same with that of the actual variation of u within that space. We assume here, that the function u varies continuously within the space ω'; in other words, that if ω' be divided into any number of equal parts, the variations which u receives in each of these parts are quantities of the same order of magnitude. Hence it is easily seen, that the variation of u within the space $k''e^{i''}$ will be represented by an expression of the form

$$k'''e^{i''-i'}.$$

For if ω' be divided into a number of parts, each equal to $k''e^{i''}$, the variations in these segments may be represented by

$$a_1e^m,\ a_2e^m,\ \&c.,$$

the exponent m being, in conformity with the foregoing assumption, the same for all. Hence we shall have for the complete variation of u,

$$(a_1 + a_2 + \&c.)\, e^m.$$

Now since a_1, a_2, &c., are finite quantities,

$$a_1 + a_2 + \&c.,$$

will be a quantity of the same order as their number. Denoting this number by n, we shall have

$$n = \frac{\omega'}{k''e^{i''}} = \frac{k'}{k''}\, e^{i'-i''}.$$

Hence it is evident, that the complete variation of u is of the order

$$m + i' - i''.$$

Since, therefore, this variation is by hypothesis finite, we must have

$$m + i' - i'' = 0,$$

or

$$m = i'' - i'.$$

Hence the expression for the partial variation of u is

$$k'''e^{i''-i'}.$$

Let u_1 be the least value of u within the element v, and $u_1 + k'''e^{i''-i'}$ the greatest, and let it be supposed, as the most unfavourable case, that u has throughout the value

$$u_1 + k'''e^{i''-i'}.$$

Substituting this expression in $\Sigma_1 mu$ we have

$$\Sigma_1 mu = u_1 \Sigma_1 m + k'''e^{i''-i'} \Sigma_1 m.$$

Hence the error in the equation

$$\Sigma_1 mu = u_1 \Sigma_1 m$$

is at most

$$k'''e^{i''-i'} \Sigma_1 m;$$

and, therefore, the error in

$$\Sigma mu = u_1 \Sigma_1 m + u_2 \Sigma_2 m + \&c.,$$

is, at most, a quantity of the form

$$Ke^{i''-i'}.$$

This equation will, therefore, be free from sensible error if

$$i'' > i'.$$

(2.) In estimating the error produced by the second supposition, we shall assume that the densities and magnitudes of the molecules vary with a *finite* degree of rapidity; and that, therefore, at any one point in the body, the sum of the masses of the molecules contained in an element is proportional to their number. Hence the equation

$$\Sigma_1 m = \epsilon_1 v,$$

is equivalent to an assumption, that the number of molecules contained in the element v is proportional to its volume.

To estimate the error involved in this assumption, let us compare, for the sake of greater generality, two elements whose bounding surfaces are wholly different in form. Suppose these elements to be similarly divided into rectangular prisms with the same transverse section, whose linear dimension is of the same order with the molecular distance. The error involved in such

a supposition will be, for each of the prisms, represented by the expression

$$Ae^{3i};$$

and therefore, for the whole element, by

$$Ae^{3i} \times \text{number of prisms}.$$

But the number of these prisms, being directly as the volume of the element and inversely as the volume of each prism, will be represented by the expression

$$Be^{2(i'-i)}.$$

Hence the error in the foregoing division will be for each element

$$Ce^{2i''+i}.$$

Now since the bounding planes of these prisms, taken two and two, are symmetrically situated with regard to the molecules which they contain, if the extremities of the prisms were symmetrically situated with regard to the extreme molecules, the number of such molecules contained in these two prisms would evidently be as their lengths. But these extremities can always be made symmetrical by adding to one of the prisms a portion whose length is of the same order as the molecular distance. Hence the error involved in the assumption, that the number of molecules in each prism is proportional to its length, is represented by an expression similar to

$$Be^{2(i''-i)}.$$

The total error for each element is, therefore, expressed by a quantity similar to $Ce^{2i''+i}$. Let l, l', l'', &c., be the lengths of the several prisms into which the element v is divided, and let $\lambda, \lambda', \lambda''$, &c., denote the corresponding quantities for v'. Let also ω be the common transverse section. Then it follows from the assumptions which we have made, that

$$\Sigma_1 m = E\,(l + l' + l'' + \&c.),$$
$$\Sigma_2 m = E'(\lambda + \lambda' + \lambda'' + \&c.).$$

We have also

$$v = \omega\,(l + l' + l'' + \&c.),$$
$$v' = \omega\,(\lambda + \lambda' + \lambda'' + \&c.),$$

and therefore,

$$\frac{\Sigma_1 m}{\Sigma_2 m} = \frac{E}{E'} \cdot \frac{v}{v'}.$$

Hence, in general,

$$\Sigma_1 m = \epsilon_1 v.$$

Now we have seen that the error involved in the supposition from which this equation is derived, is for each element represented by an expression of the form

$$Ce^{3i''+i}.$$

The order of the total error will be found by multiplying this expression by the number of the elements. Now

$$\text{Number of elements} = \frac{\text{total mass}}{v} = \frac{M}{k''} e^{-3i''}.$$

Hence the order of the total error will be

$$i - i''.$$

The equation

$$\Sigma_1 m = \epsilon_1 v$$

will, therefore, be free from sensible error if

$$i > i''.$$

(3.) Lastly, it is easily shown, by reasoning similar to that of (1) and (2), that the error in the equation

$$\Sigma u\epsilon v = \iiint u\epsilon dv,$$

is at most of the order i''. The method here employed will, therefore, be free from sensible error if the three following equations hold:

$$i'' - i' > 0, \quad i - i'' > 0, \quad i'' > 0.$$

Hence we infer that

The methods of the integral calculus are applicable to questions of molecular mechanics, provided that the molecular force varies continuously within its sphere of action; and provided also that the sphere of molecular action is of such a magnitude as to admit of being subdivided into an indefinite number of elements, each element containing an indefinite number of molecules.

NOTE ON ARTICLE 12.

ON a reperusal of Article 12, it appears to me that I have not stated with sufficient accuracy the distinguishing characteristic of a system of attracting and repelling molecules, and I think it, therefore, necessary to add a few words in explanation of what I have there said.

The supposition that the molecular force is a function of the distance may have one of two meanings, namely:—1. That all molecules situated at the same distance from m exert upon it a force of the same intensity. 2. That the force which any one molecule m' exerts upon m cannot be changed, except by altering the distance between these two molecules. If we recollect that the symbol d denotes a passage from one molecule to another, and δ the displacement of the *same* molecule; and if we use the latter in its most general sense, as applied to any displacement, virtual or real, we may represent the first hypothesis by the equations

$$\frac{dF}{d\theta} = 0, \quad \frac{dF}{d\phi} = 0;$$

and the second by

$$\frac{\delta F}{\delta\theta} = 0, \quad \frac{\delta F}{\delta\phi} = 0.$$

It is in the latter of these significations that the second hypothesis made in Article 12 is to be understood.

Now it has been shown in the text, that if the sum of the internal moments be capable of being represented by the variation of a single function, we must have

$$F_1 = f(\rho') = a\rho',$$

since ρ' is indefinitely small. If then we use the symbol δ to denote an increment produced by a *real* displacement, we shall have

$$F_1 = \delta F, \quad \rho' = \delta\rho,$$

and, therefore,

$$\delta F = a\delta\rho.$$

Hence

$$\frac{\delta F}{\delta\theta} = 0, \quad \frac{\delta F}{\delta\phi} = 0,$$

denoting, as we have seen, that the molecular force is, in the sense above defined, a force of attraction or repulsion. If, however, we confine ourselves to the strict sense of the terms attraction and repulsion, we ought to define the force solely by its *direction*, without any regard to its intensity. Properly speaking, therefore, the equations of motion of a system of attracting or repelling molecules will, in their most general form, contain thirty distinct constants.